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Classification of $\mathcal{N} = 2$ supersymmetric CFT$_4$s: indefinite series

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Abstract

Using the geometric engineering method of 4D $\mathcal{N} = 2$ quiver gauge theories and results on the classification of Kac–Moody (KM) algebras, we show by explicit examples that there exist three sectors of $\mathcal{N} = 2$ infrared CFT$_4$s. Since the geometric engineering of these CFT$_4$s involves type II strings on K3 fibred CY3 singularities, we conjecture the existence of three kinds of singular complex surfaces containing, in addition to the two standard classes, a third indefinite set. To illustrate this hypothesis, we give explicit examples of K3 surfaces with H_3^4 and E_{10} hyperbolic singularities. We also derive a hierarchy of indefinite complex algebraic geometries based on affine A_r and $\mathrm{T}_{(p,q,r)}$ algebras going beyond the hyperbolic subset. Such hierarchical surfaces have a remarkable signature that is manifested by the presence of poles.

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1. Introduction

Recently D-dimensional supersymmetric conformal field theories (CFT$_D$) have been subject to an intensive interest in connection with superstring compactifications on Calabi–Yau (CY) manifolds [1–4] and AdS/CFT correspondence [5, 6]. An important class of these super CFTs corresponds to those embedded in type II string compactifications on K3 fibred CY threefolds (CY3) with *ADE* singularities. These theories admit a very nice geometric engineering [7, 8] in terms of quiver diagrams and are classified into two categories according to the type of K3 singularities: (a) $\mathcal{N} = 2$ CFT$_4$s with gauge group $G = \prod_i SU(s_i n)$ and bi-fundamental matters. This category of scale invariant field models is classified by *affine* $\widehat{ADE}$ Lie algebras. They have vanishing individual beta function b_i known to be given by

$b_i = \frac{1}{12}\big(44n_i - \sum_j \big[8a_{ij}^4 + 2a_{ij}^6\big]n_j\big)$ with a_{ij}^4 and a_{ij}^6 being the numbers of Weyl fermions and scalars respectively [2, 9]. In $\mathcal{N} = 2$ affine CFT$_4$s, this beta function relation can be put in the form $b_i = \frac{11}{6}\mathbb{K}_{ij}^{(0)}n_j$ and its vanishing condition $\mathbb{K}_{ij}n_j = 0$ can be solved in terms of the usual Dynkin integer weights $s_i(\mathbb{K}_{ij}s_j = 0)$ as follows,

$$\mathbb{K}_{ij}^{(0)}n_j = n\mathbb{K}_{ij}^{(0)}s_j = 0, \tag{1}$$

where $\mathbb{K}_{ij}^{(0)}$ is the affine $\widehat{ADE}$ Cartan matrix. The extra upper index on $\mathbb{K}_{ij}^{(0)}$ is introduced for later use. (b) $\mathcal{N} = 2$ CFT$_4$s, based on *finite ADE* singularities; with gauge group $G = \prod_i SU(n_i)$ and matters in both fundamental $\mathbf{n}_i$ and bi-fundamental $(\mathbf{n}_i, \bar{\mathbf{n}}_j)$ representations of G. In this case, the beta function b_i may be put in the form $b_i = \frac{11}{6}\big(\mathbb{K}_{ij}^{(+)}n_j - m_i\big)$ and so its vanishing condition is equivalent to

$$\mathbb{K}_{ij}^{(+)}n_j = +m_i, \tag{2}$$

where now $\mathbb{K}_{ij}^{(+)}$ is the finite *ADE* Cartan matrix and where m_i is interpreted as the number of fundamental matters. Here also, we have introduced the extra upper index on $\mathbb{K}_{ij}^{(+)}$ to distinguish it from $\mathbb{K}_{ij}^{(0)}$ of equation (1). Note that equation (2) may be thought of as a special deformation of equation (1), which in field-theoretic language consists in adding a definite number of Weyl fermions and scalars; that is, more supersymmetric fundamental matters. This interpretation is not a new idea in QFT$_d$; something close to that was already used in the study of deformations of the 2D conformal structure; in particular in the analysis of deformations of 2D Toda field theories. In the present 4D case, much information on the deformation of equations (1), (2) and vice versa may be read directly on the explicit relation $b_i = \frac{1}{12}(44n_i - \vartheta_i)$ with $\vartheta_i = \sum_j \big[8a_{ij}^4 + 2a_{ij}^6\big]n_j$. Starting from $b_i > 0$, that is $44n_i > \vartheta_i$, one can recover conformal invariance by adding appropriate amount of fundamental matter to the quiver gauge system; this corresponds to increasing ϑ_i until the conformal point is reached. Pushing this reasoning further by remarking that as one may *add* matter, one may also *integrate it out*. This corresponds to starting from $b_i < 0$, i.e. $44n_i < \vartheta_i$ and integrating out some amount of matter which decreases ϑ_i. The resulting beta function can be put in the form $\frac{11}{6}\big(\mathbb{K}_{ij}^{(-)}n_j + m_i\big)$; so one ends with the following conformal invariant dual formula to equation (2),

$$\mathbb{K}_{ij}^{(-)}n_j = -m_i, \qquad i = 1, \ldots. \tag{3}$$

To give an interpretation to $\mathbb{K}_{ij}^{(-)}$ matrix, note that the above three equations show that they are really very remarkable relations in the sense that they may be put altogether into a condensed form as follows:

$$\mathbb{K}_{ij}^{(q)}n_j = qm_i, \qquad q = +1, 0, -1. \tag{4}$$

But this formula is very well known in the literature on KM algebras as it is just the statement of the theorem of their classification which says that the three $q = +1, 0, -1$ sectors correspond respectively to finite, affine and indefinite classes of KM algebras [10].

In this paper, we develop the study for the particular class of indefinite $\mathcal{N} = 2$ CFT$_4$s. We will show that this class shares all the basic features we know about finite and affine $\mathcal{N} = 2$ QFT$_4$s and their IR CFT$_4$ limits embedded in type II string on CY3 with singular K3 fibration. As a consequence of this classification, we conjecture the existence of a third class of local K3s with indefinite singularities; the two others are the known *ADE* ones. As we usually do in finite and affine standard cases, we will focus our attention here also on the simply laced subset of local K3s classified by indefinite KM algebras and the corresponding mirror geometries. More precisely, we study the special case of $\mathcal{N} = 2$ CFT$_4$ models based on simply laced hyperbolic symmetries as well as particular extensions.

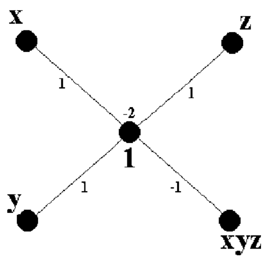

Figure 1. A typical trivalent vertex in mirror geometry. It involves a central node and four attached ones; two of them are of Dynkin type and the others are required by CY condition. They deal with fundamental matters.

The presentation of this paper is as follows: in section 2, we review briefly the computation of the general expression of the beta function of $\mathcal{N} = 2$ QFT$_4$s using the geometric engineering method. Then, we show that the solution for $\mathcal{N} = 2$ CFT$_4$ scale invariance condition coincides exactly with the Lie algebraic classification equation (4). In sections 3 and 4, we establish a classification theorem for $\mathcal{N} = 2$ CFT$_4$s and give two explicit illustrating examples. These concern local K3 with hyperbolic H_3^4 and E_{10} singularities. In section 5, we give a conclusion and generalizations.

2. Beta function in $\mathcal{N}$ = 2 quiver QFT$_4$

A nice way to compute the beta function of the $\mathcal{N} = 2$ quiver gauge theories is to use the geometric engineering method of QFT$_4$s embedded in type II strings on CY3 with *ADE* singularities [7]. This method involves toric representation of CY3, mirror symmetry and techniques of algebraic geometry; in particular trivalent geometry, main lines of which we review here. Details can be found in [7, 8]. To illustrate the idea of the method in a comprehensive way, we start by considering the case of a unique trivalent vertex; then we give the results for chains of trivalent vertices.

Case of one trivalent vertex. In type IIA string on CY3, a typical trivalent vertex of the toric representation of CY3 is described by the three-dimensional vectors V_i,

$$\begin{aligned} &V_0 = (0, 0, 0), \qquad V_1 = (1, 0, 0), \qquad V_2 = (0, 1, 0), \\ &V_3 = (0, 0, 1), \qquad V_4 = (1, 1, 1) \end{aligned} \tag{5}$$

satisfying the following toric geometry relation $\sum_{i=0}^{4} q_i V_i = -2V_0 + V_1 + V_2 + V_3 - V_4 = 0$. The vector charge $(q_i) = (-2, 1, 1, 1, -1)$ is known as the Mori vector and the sum of its q_i components is zero as required by the CY condition. In type IIB mirror geometry, the $(V_0, V_1, V_2, V_3, V_4)$ vertices are represented by complex variables $(u_0, u_1, u_2, u_3, u_4)$ constrained as $\prod_i u_i^{q_i} = 1$ and solved by $(1, x, y, z, xyz)$ (see figure 1). In terms of these variables, the algebraic geometry equation describing mirror geometry is given by the following complex surface, $P(X^*) = e_0 + a_0 x + b_0 y + (c_0 - d_0 xy)z$, where a_0, b_0, c_0, d_0 and e_0 are non-zero complex moduli. Upon eliminating the z variable by using the equation of motion $\frac{\partial P(X^*)}{\partial z} = 0$, the above trivalent geometry reduces exactly to

$$P(X^*) = a_0 x + e_0 + \frac{b_0 c_0}{d_0} \frac{1}{x}, \tag{6}$$

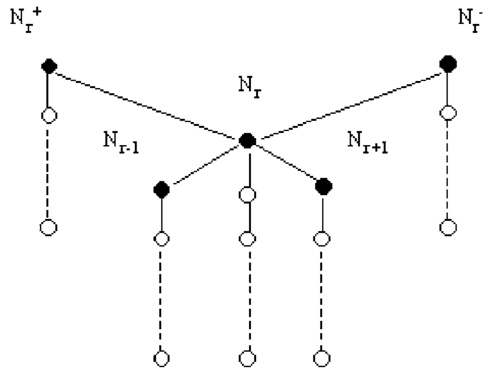

Figure 2. This graph describes a typical vertex one has in geometric engineering of $\mathcal{N} = 2$ QFT_4. $SU(1 + l)$ gauge and flavour symmetries are fibred over the five black nodes. Flavour symmetries require large base volume.

which is nothing but the mirror of the su(2) singularity of local K3 surface. To get the equation of the CY3, one promotes the coefficients a_0, b_0, c_0, d_0 and e_0 to holomorphic polynomials on complex plane as

$$\begin{aligned} e &= \sum_{i=0}^{n_r} e_i \zeta^i, \qquad a = \sum_{i=0}^{n_{r-1}} a_i \zeta^i, \qquad b = \sum_{i=0}^{n_{r+1}} b_i \zeta^i, \\ c &= \sum_{i=0}^{m_r} c_i \zeta^i, \qquad d = \sum_{i=0}^{m'_r} d_i \zeta^i. \end{aligned} \tag{7}$$

Note that the functions a, b and e encode the fibrations of $SU(1 + n_{r-1}) \times SU(1 + n_r) \times SU(1 + n_{r+1})$ gauge symmetry while c and d are associated with flavour symmetries of the underlying $\mathcal{N} = 2$ QFT_4 engineered over the nodes of the trivalent vertex. The nature of the flavour group will be discussed later on; all what we know about it is that for $m'_r = 0$, the group is $SU(1 + m_r)$ but this corresponds to a finite class of $\mathcal{N} = 2$ CFT_4s. Note also that in the geometric engineering method, the $SU(1 + n_r)$ and $SU(1 + n_{r\pm1})$ gauge symmetries are fibred over V_0, V_1 and V_2. However the two kinds of 'matters' m_r and m'_r are fibred over the nodes V_3 and V_4 respectively (see figure 2). Note finally that all of the holomorphic functions a, b, c, d and e are not independent; one can usually fix one of them. We will see that this freedom turns into a condition on m_r and m'_r; but for the moment, we keep all these moduli free and make a comment later on.

Infrared $\mathcal{N} = 2$ QFT_4 limit. To get the various $\mathcal{N} = 2$ CFT_4s embedded in type IIA strings on CY3, we have to study the infrared field theory limit one gets from mirror geometry equation (6) and look for the scaling properties of the gauge coupling constant moduli. We will do this explicitly for the case of the trivalent vertex and then give the general result for the chain. To that purpose, we proceed in three steps: first determine the behaviour of the complex moduli f_i appearing in expansion equation (7) under a shift of ζ by $1/\varepsilon$ with $\varepsilon \to 0$. Doing this and requiring that equations (7) should be preserved, which is still staying in the singularity described by equations (7), we get the following,

$$e_l \sim \varepsilon^{l-n_r}, \qquad a_l \sim \varepsilon^{l-n_{r-1}}, \qquad b_l \sim \varepsilon^{l-n_{r+1}}, \qquad c_l \sim \varepsilon^{l-m_r}, \qquad d_l \sim \varepsilon^{l-m'_r}. \tag{8}$$

Second, compute the scaling behaviour of the gauge coupling constant moduli $Z^{(g)}$ under the shift $\zeta' = \zeta + 1/\varepsilon$. Putting equations (8) back into the explicit expression of $Z^{(g)}$ namely $Z^{(g)} = \frac{a_0 b_0 c_0}{e_0^2 d_0}$, we get the following behaviour $Z^{(g_r)} \sim \varepsilon^{-b_r}$ with b_r given by

$$b_r = \tfrac{11}{6}[2n_r - n_{r-1} - n_{r+1} - (m_r - m'_r)]. \tag{9}$$

This relation tells us: (i) b_r is the beta function for the gauge group factor $SU(1 + n_r)$. (ii) b_r depends on $m_i^* = m_r - m'_r$; it is invariant under global shifts of m_r and m'_r, a property which reflects the arbitrariness we have referred to above. Introducing the following notation sing $(m_i^*) = q$ with $q = +1, 0, -1$ respectively associated with the intervals $m_r > m'_r$, $m_r = m'_r$ and $m_r < m'_r$, we can rewrite equation (9) as $\mathbf{K}_{ij}^{(q)} n_j - q|m_i^*|$ (see also equation (4)). Finally taking the limit $\varepsilon \to 0$, finiteness of $Z^{(g)}$ requires then that the field theory limit should be asymptotically free; that is $b_r \leqslant 0$. Upper bound $b_r = 0$ corresponds to the scale invariance we are interested in here.

Conformal invariance phases. From equation (9) it is not difficult to recognize the three classes of solutions for $\mathbf{K}_{ij}^{(q)} n_j = q m_i^*$: (i) $m_r - m'_r = 0$ and $n_r = n_{r-1} = n_{r+1} = n$; this corresponds to a generic vertex of $\widehat{SU(k)}$ affine $\mathcal{N} = 2$ conformal CFT$_4$ with $SU(n)^3$ gauge symmetry. Extension to the other $\widehat{DE}$ geometries is straightforward. (ii) $m'_r = 0$, but the other integers may be taken as $n_r = \alpha n; n_{r-1} = \beta n, n_{r+1} = \gamma n, m_r = \delta n$ with $\alpha, \beta, \gamma, \delta \in n\mathbb{Z}_+$ constrained as $2\alpha = \beta + \gamma + \delta$. As an example, one may take them as $m_r = n_{r-1} = n_{r+1} = 2n$ and $n_r = 3n$; this corresponds to a gauge symmetry $SU(3n) \times SU(2n)^2$ and an $SU(2n)$ flavour symmetry engineered on the middle vertex of the $SU(4)$ finite Dynkin diagrams. This solution is also valid for $m_r - m'_r > 0$; all one has to do is to substitute the expression of m_r of the above solution by m_r^*. (iii) For the remarkable case $m_r = 0$; that is $m_r^* < 0$, conformal invariance requires $2n_r - n_{r-1} - n_{r+1} + m'_r = 0$ and is solved as $n_r = \alpha n; n_{r-1} = \beta n, n_{r+1} = \gamma n, m'_r = \delta' n$ with $\alpha, \beta, \gamma, \delta' \in n\mathbb{Z}_+$ satisfying $2\alpha + \delta' = \beta + \gamma$. As an example, one may take them as $m'_r = n_{r-1} = n_{r+1} = 2n$ and $n_r = n$. Note that solutions for conformal invariance may have $m'_r > n_r$ as one sees on the above particular solution. This property constitutes one of the arguments we will use to conjecture the flavour symmetry $SU(qm_r^*)$; it recovers the known results as particular cases. Naturally the $q = -1$ sector corresponds to a new class of solutions. In this regard we will show that this class is linked with simply laced indefinite KM algebras. To do so we need however more than one trivalent vertex since simply laced indefinite Lie algebras have at least a rank four and this corresponds to the overextension of affine $\widehat{A}_2$.

Chains of trivalent vertices. To get the generalization of the above results, it is enough to think about the previous vertices as a generic trivalent vertex of a linear chain of N trivalent vertices, that is

$$V_0 \to V_\alpha^0, \qquad V_3 \to V_\alpha^+, \qquad V_4 \to V_\alpha^-, \qquad V_1 \to V_{\alpha-1}^0, \qquad V_2^0 \to V_{\alpha+1}^0, \tag{10}$$

where $\alpha \in \{1, \ldots, N\}$. The intersections between V_α^0 and $V_{\alpha\pm1}^0$ are specified by some integers q_α^i generally inspired from the Cartan matrix of the KM algebra one is interested in. In this generic case, the data of the toric polytope are fixed by $\sum_{\alpha\geqslant 0} \left(\mathrm{q}_\alpha^i V_\alpha^0 + V_i^+ - V_i^-\right) = 0$ and $\sum_\alpha^i q_\alpha^i = 0$. Note that the $\pm$ upper indices carried by the $V_i^\pm$ vertices refer to the fourth +1 and five -1 entries of the Mori vector $q_\tau^i = \left(q_\alpha^i; +1, -1\right)$ of trivalent vertex. In practice, the Mori vectors q_α^is form a $N \times (N + s)$ rectangular matrix whose $N \times N$ square sub-matrix q_j^i is minus the generalized Cartan matrix $\mathbb{K}_{ij}^{(q)}$. For the example of affine A_{N-1}, the Mori charges read as $q_\alpha^i = 2\delta_\alpha^i - \delta_\alpha^{i-1} - \delta_\alpha^{i+1}$ with the usual periodicity of affine $\widehat{SU(n)}$. The remaining $N \times s$ part of q_α^i is fixed by the CY condition $\sum_\alpha^i q_\alpha = 0$ and the corresponding vertices are interpreted as dealing with non-compact two-dimensional divisors defining the singular space on which singularities lie. In mirror geometry where $x_{\alpha-1}, x_\alpha, x_{\alpha+1}, y_\alpha$ and $\frac{x_{\alpha-1}x_{\alpha+1}y_\alpha}{y_\alpha^2}$ are the variables associated with the vertices (10), algebraic equation for a generic vertex extends as $a_{\alpha-1}x_{\alpha-1} + a_\alpha x_\alpha + a_{\alpha+1}x_{\alpha+1} + c_\alpha y_\alpha + d_\alpha \frac{x_{\alpha-1}x_{\alpha+1}y_\alpha}{x_\alpha^2} = 0$ where a_α, c_α and d_α are complex moduli. Summing over the vertices and setting $y_\alpha = x_\alpha z_\alpha$, one gets $P(X^*) = a_0 x_0 + \sum_{\alpha\geqslant 1}\left(a_\alpha x_\alpha + c_\alpha x_\alpha z_\alpha + d_\alpha \frac{x_{\alpha-1}x_{\alpha+1}z_\alpha}{x_\alpha}\right)$. Eliminating the variable z_α as we have done for equation (6), we obtain

$$P(X^*) = \sum_{\alpha\geqslant 0} x^\alpha a_\alpha(w) \prod_{\beta\geqslant 1}\left(\frac{\mathrm{c}_\beta(w)}{d_\beta(w)}\right)^{\alpha-\beta}. \tag{11}$$

From this relation, one gets behaviour $Z^{(g_r)} \sim \varepsilon^{-b_r}$ with b_r given by

$$b_r^{(q)} = \tfrac{11}{6}[2n_r - n_{r-1} - n_{r+1} - q|m_r^*|], \qquad r = 1, \ldots. \tag{12}$$

3. Classification theorem of $\mathcal{N} = 2$ CFT$_4$s

Let $\mathcal{G}_q$ be some given *simply laced* Lie algebra of rank $\mathrm{r}_q = \mathrm{rank}(\mathcal{G}_q)$ and Cartan matrix $\mathbb{K}^{(q)}$, $\mathrm{corank}(\mathbb{K}^{(q)}) \leqslant 1$ and let $q = +1, 0$ and -1 be an integer which refers respectively to the three possible sectors of $\mathcal{G}_q$ that are of *finite, affine and indefinite* types. Then the previous results on $\mathcal{N} = 2$ quiver gauge CFT$_4$s can be stated as a theorem to which we shall refer hereafter as the classification theorem of $\mathcal{N} = 2$ CFT$_4$s. As these supersymmetric gauge theories are special limits of underlying 4D massive field theories (QFT$_4$), we will state this theorem in a more general way.

Theorem. *For any quiver graph $\Delta(\mathcal{G}_q)$ of trivalent vertices with a topology-type Dynkin diagram of the simply laced (finite, affine and indefinite) Lie algebras $\mathcal{G}_q$, there corresponds:*

(a) A $\mathcal{N} = 2$ quiver gauge QFT$_4$s which is built as usual by extending the geometric engineering method to include indefinite type Dynkin diagrams. They may be denoted as QFT$_4^{(q)}$.

(b) The quiver gauge group of these $\mathcal{N} = 2$ QFT$_4^{(q)}$s is $\prod_{i=1}^{\mathrm{r}_q} SU(n_i)$ and the flavour symmetry encoding fundamental matters read as $\prod_{i=1}^{\mathrm{r}_q} SU(qm_i^*)$. Here, the positive integer $|m_i^*|$ is the effective number of fundamental matters that contribute to the beta function; it depends on the absolute value of the difference of m_i and m_i'.

(c) The b_r functions of the $SU(n_i)$ gauge symmetries of these $\mathcal{N} = 2$ quiver QFT$_4$s read as,

$$b_r^{(q)} = \tfrac{11}{6}\left(\mathbb{K}_{rs}^{(q)} n_s - q|m_r^*|\right), \qquad r = 1, 2, \ldots, r_q \tag{13}$$

where q refers to the three above-mentioned sectors.

(d) In the infrared limit of $\mathcal{N} = 2$ gauge quiver QFT$_4$s where $b_r^{(q)} \longrightarrow 0$, these theories flow to three classes of 4D $\mathcal{N} = 2$ quiver conformal field theories. The flows are in one-to-one correspondence with the three sectors of $\mathcal{G}_q$s. As such $\mathcal{N} = 2$ CFT$_4$s are classified as QFT$_4^{(q)}$:

(i) Finite ADE $\mathcal{N} = 2$ CFT$_4^+$*s for which the vanishing of the beta function leads to $\mathbb{K}_{rs}^{(+)} n_s = |m_r^*|$.*

(ii) Affine ADE $\mathcal{N} = 2$ *quiver* CFT$_4^0$*s governed by $\mathbb{K}_{rs}^{(0)} n_s = 0$ with one-dimensional corank $\left(\mathbb{K}_{rs}^{(0)}\right)$.*

(iii) Indefinite $\mathcal{N}=2$ quiver CFT$_4^-$ s. They are associated with the class $\mathbb{K}_{rs}^{(-)}n_s = -|m_r^*|$ where now $\mathbb{K}_{rs}^{(-)}$ is an indefinite Cartan matrix.

To prove this theorem, note that the first three properties follow naturally from the algebraic geometry analysis of the $\mathcal{N}=2$ quiver QFT$_4$s embedded in type IIA string on CY3 [7, 8] and references therein. The fourth property (d) of this theorem can be linked to the Vinberg–Kac–Moody basic theorem on the classification of Lie algebras which we recall here below. Property (d) follows from it by setting $u_i = n_i$ and $v_i = |m_r^*|$.

Vinberg–Kac–Moody theorem. A generalized indecomposable Cartan matrix $\mathbb{K}$ obeys one and only one of the following three statements: (i) *Finite type* ($\det\mathbb{K} > 0$): there exists a real positive definite vector $\mathbf{u}$ ($u_i > 0$; $i = 1, 2, \ldots$) such that $\mathbb{K}_{ij}u_j = v_j > 0$. (ii) *Affine type*, corank ($\mathbb{K}$) $= 1$, $\det\mathbb{K} = 0$: there exists a unique, up to a multiplicative factor, positive integer definite vector $\mathbf{u}$ ($u_i > 0$; $i = 1, 2, \ldots$) such that $\mathbb{K}_{ij}u_j = 0$. (iii) *Indefinite type* ($\det\mathbb{K} \leqslant 0$), corank ($\mathbb{K}$) $\neq 1$: there exists a real positive definite vector $\mathbf{u}$ ($u_i > 0$; $i = 1, 2, \ldots$) such that $\mathbb{K}_{ij}u_j = -v_i < 0$.

All the equations appearing in this theorem combine together to give equation (4). As a consequence of this classification of $\mathcal{N}=2$ CFT$_4^{(q)}$s, our theorem may also be viewed as a classification of possible K3 singularities. We have then the following.

Corollary. *From the property (d) of our classification theorem, we conjecture the existence of indefinite singularities for K3 fibred CY threefolds that are characterized by simply laced indefinite Lie algebras. With this hypothesis, we have: (α) finite ADE singularities; (β) affine $\widehat{ADE}$ singularities; (γ) indefinite singularities.*

Note that the above two first singular K3 surfaces are well common in type II strings on CY3. However the third one is a new class which to our knowledge has not been studied before. It is dictated from $\mathcal{N}=2$ field-theoretic analysis of $\mathcal{N}=2$ CFT$_4^{(q)}$ possible solutions. In [11], we have made a general analysis of such kind of singularities; here we give explicit illustrating examples. They concern the overextension of affine $\widehat{A}_2$ and the overextension of $\widehat{E}_8$ respectively denoted as H_4^3 and E_{10}.

4. Two examples of hyperbolic singularities

We begin by recalling that the mirror geometry of type IIA string on CY3 (X_3) with affine $\widehat{ADE}$ singularities is conveniently described in algebraic geometry. A typical equation of such geometry is $P(X_3^*) = \sum_\alpha a_\alpha y_\alpha$, where X_3^* is the mirror of X_3 and where $a_\alpha = a_\alpha(w)$ are complex moduli with expansion similar to those of equation (7), (see also [7]). In this relation, the y_α complex variables are constrained as,

$$\prod_{j=1}^{n} y_j^{q_j^i} = \prod_{\alpha=n+1}^{n+4} y_\alpha^{-q_\alpha^i}, \tag{14}$$

where q_j^i is minus the Cartan matrix $\mathbb{K}_{ij}$ of the corresponding Lie algebra and y_α, with $n < \alpha < n+5$, are four extra complex variables that are just the monomials appearing in the elliptic curve $E = y^2 + x^3 + z^6 + \mu xyz = 0$ on which shrinks the affine ADE singularity. Therefore, we have,

$$y_{n+1} = y^2, \qquad y_{n+2} = x^3, \qquad y_{n+3} = z^6, \qquad y_{n+4} = xyz, \tag{15}$$

where (y, x, z) are the homogeneous coordinates of the weighted projective space $\mathbf{WP}^2(3, 2, 1)$. The remaining n complex variables y_i definitive the $\widehat{ADE}$ geometry are also solved in terms of the previous y, x and z variables. Such solutions depend on the q_j^i and q_α^i integer charges forming altogether a $n \times (n+4)$ rectangular matrix as

$$Q_\alpha^i = \left(q_j^i, \quad q_{n+1}^i, \quad q_{n+2}^i, \quad q_{n+3}^i, \quad q_{n+4}^i\right). \tag{16}$$

The resulting two-dimensional geometry $y^2 + x^3 + z^6 + \mu_0 xyz + \sum_{i=1}^n a_i y_i = 0$ has been studied extensively in the literature for both trivalent and affine geometries. But here we are claiming that such analysis applies as well for the indefinite sector of Lie algebras and deals with the un-explored class of indefinite CFT_4s. As the best way to justify our claim is to give examples, we will start by recalling some useful features on affine geometries and then study the indefinite case. Before note that the parameter μ appearing in the algebraic geometry equation of the elliptic curve $E(\mu)$ is its complex structure. It is fixed to a constant μ_0 in the case of affine *ADE* geometries; but varies in the case indefinite singularities we are interested in here. More precisely, we will see that in the case of simply laced hyperbolic geometries, the parameter μ has to vary on a complex plane parametrized by w; i.e. $\mu = \mu(w)$. Under this variation, the initial curve $E(\mu_0)$ is now promoted to a complex surface $E[\mu(w)]$ which, by the way, is nothing but the elliptic fibration of K3 $y^2+x^3+z^6+\mu(w)xyz = 0$. Note that, upon appropriate redefinition of variables, one may rewrite the above algebraic geometry equation of the elliptic curve into the following equivalent form,

$$y^2 + x^3 + \nu(t)z'^6 + xyz' = 0 \tag{17}$$

where now $z' = \mu(w)z$ and $\nu(t)z'^6 = z^6$. For instance, if we take $\nu(t) = t^{-1} = w^{-6}$, then z' should be $z' = wz$ and so $\mu(w) = w$. Having these properties in mind, we now turn to illustrate the building of affine A_2 geometry and its hyperbolic overextension.

Affine extension of A_2 geometry. In the special case of affine A_2 geometry, like all series of affine ADEs, one starts from the curve $E_0 = y^2 + x^3 + z^6 + \mu_0 xyz = 0$ of $\mathbf{WP}^2(3, 2, 1)$ with fixed complex structure and looks for algebraic geometry equation of affine A_2 geometry which reads as

$$\widehat{A}_2 : y^2 + x^3 + z^6 + \mu_0 xyz + (by_1 + cy_2 + dy_3) = 0. \tag{18}$$

Here b, c and d are complex moduli which once taken simultaneously to zero the affine A_2 geometry shrinks to the elliptic curve. To get the explicit expression of the remaining y_i gauge invariants, one has to specify the toric data for the present affine A_2 geometry and too particularly the q_j^i and q_α^i charges appearing in equation (14). These read as

$$Q(\widehat{A}_2) = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & -2 & 1 & 2 & 1 & 0 & -3 \\ 1 & 1 & -2 & 0 & 1 & 0 & -1 \end{pmatrix}. \tag{19}$$

The simplest solution one gets for the constraint equations (14) regarding y_1, y_2 and y_3 is $y_1 = z^3$, $y_2 = xz$ and $y_2 = y$. However this is not unique as there are infinitely many others depending on an extra free complex parameter υ as shown below,

$$y_1 = z^3\upsilon, \qquad y_2 = xz\upsilon, \qquad y_2 = y\upsilon, \tag{20}$$

where υ is a homogeneous complex parameter of scaling weight 3 so that (x, y, z, υ) parametrize the $\mathbf{WP}^3(3, 2, 1, 3)$. Therefore affine A_2 geometry reads as

$$\widehat{A}_2 : y^2 + x^3 + z^6 + \mu_0 xyz + \upsilon(bz^3 + cxz + dy) = 0. \tag{21}$$

From these relations, one may also write the vertices and the Mori charges of the corresponding toric polytope; these may be found in [11]. With relations (18)–(21) at hand, we are now ready to build our first example of complex surface with an indefinite singularity.

Overextension of affine $\mathbf{A}_2$ *geometry.* First of all note that the simplest overextension of $\widehat{A}_2$ LKM algebra is a simply laced indefinite Lie algebra; it is generally denoted as H_3^4 according to the classification of Wanglai Lie (see also the appendix) and belongs to the so-called hyperbolic subset. It has the following $\mathbb{K}(H_3^4)$ Cartan matrix,

$$\mathbb{K}(H_3^4) = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix},$$
$$\mathbb{Q}(H_3^4) = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 1 & 3 & 1 & 0 & -4 \\ 0 & 1 & 1 & -2 & 0 & 1 & 0 & -1 \end{pmatrix}. \tag{22}$$

$\mathbb{Q}(H_3^4)$ is the matrix of corresponding Mori vectors to be used later. To get the mirror geometry of a local K3 surface with H_3^4 singularity, we suppose the three following:

(a) As for a Lie algebra structure where H_3^4 appears as an overextension of affine A_2, we consider that hyperbolic H_3^4 geometry is also an extension of affine A_2 one. As such we conjecture that the algebraic geometry equation for H_3^4 surface reads as,

$$H_3^4 : y^2 + x^3 + \nu(t)z^6 + xyz + \left(\sum_{i=1}^{4} a_i y_i\right) = 0, \tag{23}$$

where we have considered an elliptic curve with a varying complex structure. The a_is moduli describe the complex deformation of H_3^4 singularity of the hyperbolic surface and y_is are four gauge invariants that should be solved in terms of the x, y, z and t variables.

(b) Relations (14) used for affine geometries are also valid for the simply laced indefinite Lie algebra sector. As such we have, for the present example, the following relations defining the y_i gauge invariants for H_3^4 geometry,

$$\prod_{\alpha=1}^{8} y_\alpha^{\mathbb{Q}_\alpha^i} = 1, \qquad i = 1, 2, 3, 4, \tag{24}$$

where the 4×8 rectangular matrix $\mathbb{Q}_\alpha^i$ defines the four Mori vectors associated with the hyperbolic H_3^4 geometry equation (22). The property $\sum_\alpha \mathbb{Q}_\alpha^i = 0$ reflects just the CY condition of this special local K3 surface.

(c) Once the a_i moduli encoding the complex deformation of hyperbolic H_3^4 surface are taken simultaneously to zero, the H_3^4 geometry shrinks into equation (17). This means that equation (15) should be modified as

$$y_{n+1} = y^2, \qquad y_{n+2} = x^3, \qquad y_{n+3} = t^{-1}z^6, \qquad y_{n+4} = xyz. \tag{25}$$

With these tools at hand, one can solve explicitly the remaining four y_i gauge invariants in terms of the complex variables x, y, z and t of $\mathbf{WP}^2(3, 2, 1) \times \mathbb{C}^*$. We find

$$H_3^4 : y^2 + x^3 + z^6 t^{-1} + xyz + [az^6 + btz^6 + ctxz^4 + dyz^3 t] = 0. \tag{26}$$

This is the relation we have been after; it is the mirror of a complex K3 surface with a hyperbolic H_3^4 singularity. CY3 are obtained as usual by promoting the complex moduli to polynomials depending on an extra complex variable ζ as $a_i(\zeta) = \sum_{j=1}^{n_i} a_{ij}\zeta^j$, where n_i stand for the rank of $U(n_i)$ group symmetries of underlying 4D $\mathcal{N} = 2$ quiver gauge theories that are embedded in type IIA string on the above CY3 with H_3^4 singularity. Before concluding, let us give two more comments. (i) As these kinds of unfamiliar CY manifolds look a little unusual, it is interesting to also write the solutions for the vertices of the toric polytope associated with the CY3 having a H_3^4 singularity. They read as

$$\begin{array}{lll} yxz \longleftrightarrow V_8 = (0,0,0), & y^2 \longleftrightarrow V_5 = (0,0,-1), & x^3 \longleftrightarrow V_6 = (0,-1,0), \\ t^{-1}z^6 \longleftrightarrow V_7 = (-1,2,3), & z^6 \longleftrightarrow V_1 = (0,2,3), & tz^6 \longleftrightarrow V_2 = (1,2,3), \\ txz^4 \longleftrightarrow V_4 = (1,1,2), & yz^3t \longleftrightarrow V_3 = (1,1,1). & \end{array} \tag{27}$$

Using these expressions and equation (22), it is not difficult to check that these vertices satisfy the basic toric geometry relations namely $\sum_{\alpha=0}^{8} q_\alpha^i = 0$ and $\sum_{\alpha=0}^{8} q_\alpha^i V_\alpha = 0$. (ii) The second comment is to discuss the link between affine A_2 and hyperbolic H_3^4 geometries. As noted before, H_3^4 Lie algebra is just an extension of affine A_2 and so one expects that there should be a bridge between the two corresponding geometries. This is what indeed happens. Starting from the algebraic geometry equation (21) of affine A_2, namely $y^2 + x^3 + \nu_0 z^6 + xyz + (bz^3 + cxz + dy)v$, and performing the following changes,

$$\nu_0 \longrightarrow \nu(t) = \alpha t^{-1} + a, \qquad v \longrightarrow v = tz^3, \tag{28}$$

one gets exactly the hyperbolic H_3^4 mirror geometry of equation (26). Here α and a are constants.

Hyperbolic E_{10} surface. To start recall that hyperbolic E_{10} is the simplest overextension of affine E_8. It is an indefinite KM algebra belonging to the hyperbolic subset, which in Kac notation, reads as $T_{(p,q,r)}$ with $(p,q,r) = (7,3,2)$. Its Cartan matrix $\mathbb{K}(E_{10})$ is symmetric and has a negative determinant namely $\det \mathbb{K}(E_{10}) = -1$. In 4D $\mathcal{N} = 2$ gauge theory embedded in type IIA string, one may geometrically engineer the E_{10} hyperbolic QFT_4 models and their infrared $\mathrm{CFT}_4^{(-)}$ limit by considering a local CY3 with an E_{10} singularity as outlined in the classification theorem of section 3. Here we would like to derive E_{10} geometry by using toric geometry methods and local mirror symmetry. Indeed, the hypothesis of variation of the complex structure of the elliptic curve allows us to define the hyperbolic E_{10} geometry as,

$$E_{10} : y^2 + x^3 + \nu(t)z^6 + xyz + \left(\sum_{i=1}^{10} a_i y_i\right) = 0, \tag{29}$$

where a_i are complex moduli and where the ten gauge invariant variables y_i are obtained by solving the constraint equations $\prod_{\alpha=1}^{14} y_\alpha^{\mathbb{Q}_\alpha^i} = 1$. Here $\mathbb{Q}_\alpha^i = \mathbb{Q}_\alpha^i(E_{10})$ are the Mori vectors associated with the hyperbolic E_{10} geometry. $\mathbb{Q}_\alpha^i$ is a $(10+4) \times 10$ rectangular matrix whose 10×10 square block is minus E_{10} Cartan matrix. The solution of the constraint equations $\prod_{\alpha=1}^{14} y_\alpha^{\mathbb{Q}_\alpha^i} = 1$ may be obtained without major difficulty as they share features with the product of the A_7, A_3 and A_2 singularities. Straightforward computations lead to the following projective exceptional surface,

$$y^2 + x^3 + z^7 t^{-1} + xyz + a_0 t^6 + a_1 t^4 x + a_2 t^2 x^2 + b_1 y t^4 + \sum_{s=1}^{6} c_s t^{6-s} z^s = 0, \tag{30}$$

where (y,x,z,t) are complex coordinates of $\mathrm{WP}_{(3,2,1,1)}$. Note that if the a_i, b_j and c_k complex moduli are simultaneously taken to zero, one ends with a K3 surface with a hyperbolic E_{10} singularity. Moreover promoting the a_i, b_j and c_k moduli to polynomials in an extra complex variable ζ as in equations (7), one gets a CY3 with complex deformed E_{10} singularity. The degrees of these polynomials define the rank of the gauge quiver group factors, in agreement with our classification theorem of $\mathcal{N} = 2$ CFT$_4^{(-)}$s. At the end of this study, note that, as for $\widehat{A}_2$ and more generally $\widehat{A}_r$, one may here also define a hierarchy of exceptional geometries; but here these correspond just to the geometries associated with the so-called $\mathrm{T}_{(p,q,r)}$ KM algebra. Therefore, this kind of algebraic geometric hierarchies are classified by three positive integers p, q and r and the corresponding surfaces are given by,

$$(y^r t^{6-3r} + x^q t^{6-2q} + z^p t^{6-p} + xyz) + a_0 t^6 + \sum_{s=1}^{p-1} c_s t^{6-s} z^s + \sum_{s=1}^{q-1} a_s t^{6-2s} x^s + \sum_{s=1}^{r-1} b_s y^r t^{6-3r} = 0, \tag{31}$$

where as before (y, x, z, t) are in $\mathrm{WP}_{(3,2,1,1)}$. From this relation, one may re-discover known geometries obtained in earlier literature on 4D $\mathcal{N} = 2$ quiver gauge theories. Particular examples are those associated with finite D_r, finite E_s and affine E_s exceptional geometries. These three classes of geometries correspond to those $\mathrm{T}_{(p,q,r)}$ algebras with positive determinant of the Cartan matrices as shown below,

$$\det(\mathbb{K}[T_{(p,q,r)}]) = pq + pr + qr - pqr \geqslant 0. \tag{32}$$

The remaining subset of $T_{(p,q,r)}$ algebras with $\det(\mathbb{K}[T_{(p,q,r)}]) < 0$ corresponds effectively to indefinite geometries; they are described by the rational number $c = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. The complex projective surfaces with $c \geqslant 1$ are effectively those given by equation (32).

5. Conclusion

In this paper, we have shown on explicit examples that the beta function $b_i^{(q)}$ of $\mathcal{N} = 2$ quiver gauge theories carries an extra index $q = 1, 0, -1$, equation (13). In the infrared limit, these gauge theories flow to *three different* IR points and so one concludes that there exist in general three sectors of $\mathcal{N} = 2$ CFT$_4$s embedded in type IIA superstring on local CY3s. These sectors are in one to one with the three classes (finite, affine and indefinite) of simply laced KM algebras. Moreover, as these supersymmetric QFT$_4$s and their CFT$_4$ IR limits are linked with singularities of K3 fibred CY3, we have conjectured the existence of three kinds of local K3 surfaces classified by generalized Cartan matrices; one of them has indefinite singularities and the two others are the well-known ones. To illustrate this claim, we have given two explicit examples, namely singular surfaces having hyperbolic H_3^4 and E_{10} degeneracies; also known as the overextensions of affine A_2 and affine E_8 respectively. These are given by equations (26) and (30). Among our results, we have also found that hyperbolic geometries may be deduced from the affine category by varying the complex structure of the elliptic curve on $\mathbb{C}^*$ (see equations (28)). Extending this idea, we have shown that the above hyperbolic singularities are, in fact, just leading elements of a hierarchy of a subset of indefinite singular K3 surfaces obtained by iterative mechanism. For the case of affine A_2 geometry (21) for instance, one gets upon using equations (28), the following surface with deformed hyperbolic H_3^4 singularity,

$$H_3^4 : y^2 + x^3 + z^6 t^{-1} + xyz + [az^6 + btz^6 + ctxz^4 + dyz^3 t] = 0. \tag{33}$$

Repeating this procedure once more, one gets the following singular surface $y^2 + x^3 + z^6 t^{-2} + xyz + [a_{-1} z^6 t^{-1} + az^6 + btz^6 + ctxz^4 + dyz^3 t] = 0$. It is classified by the following indefinite Lie algebra of minus generalized Cartan matrix given by,

$$\mathbb{K}\big(H_{3,1}^4\big) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}, \tag{34}$$

where $H^4_{3,1}$ stands for the overextension of H^4_3. Here also, one can write the data of this toric manifold as in equations (22), (27). By successive iterations, one may further generalize this result by constructing the following hierarchy of geometries based on affine $\widehat{A}_2$,

$$\widehat{A}_{2,k} : y^2 + x^3 + z^6 t^{-k} + xyz + \sum_{s=1}^{k-1} a_{-s} t^{-s} z^6 + [az^6 + btz^6 + ctxz^4 + dyz^3 t] = 0, \qquad k = 1, \ldots, \tag{35}$$

where $\widehat{A}_{2,0}$ and $\widehat{A}_{2,1}$ stand respectively for affine $\widehat{A}_2$ and H^4_3, and $\widehat{A}_{2,k}$ with $k > 1$ refer to the other hierarchical geometries. As for $T_{(p,q,r)}$ hierarchies we have studied here, see equation (31), relations (35) have also poles in t. This is a signature of indefinite geometries.

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Appendix. Indefinite Lie algebras

Indefinite Lie algebras are still a mathematical open subject since their classification has not yet been achieved. A subset of these indefinite algebras that is quite well understood includes those known as *hyperbolic* Lie algebras [10, 12]. According to the Wanglai–Li classification, there are 238 containing the following special list of simply laced ones:

$$\begin{array}{ccccccccc} \mathcal{H}^4_1, & \mathcal{H}^4_2, & \mathcal{H}^4_3, & \mathcal{H}^5_1, & \mathcal{H}^5_8, & \mathcal{H}^6_1, & \mathcal{H}^6_5, & \mathcal{H}^6_6, & \mathcal{H}^7_1, \\ \mathcal{H}^7_1, & \mathcal{H}^8_1, & \mathcal{H}^8_4, & \mathcal{H}^8_5, & \mathcal{H}^9_1, & \mathcal{H}^9_4, & \mathcal{H}^9_5, & \mathcal{H}^{10}_1, & \mathcal{H}^{10}_4. \end{array} \tag{36}$$

For other applications of hyperbolic Lie algebras in string theory, see [13, 14] and references therein.

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